2.1. Levi-Civita connection on a submanifold.

Let $(\bar{M}, \bar{g})$ be a Riemannian manifold with Levi-Civita connection $\bar{D}$, and let M be a submanifold of $\bar{M}$, equipped with the induced metric $g := i^*\bar{g}$, where $i\colon M \to \bar{M}$ is the inclusion map. Show that the Levi-Civita connection D of (M, g) satisfies $D_X Y = (\bar{D}_X Y)^{\mathrm{T}}$ for all $X, Y \in \Gamma(TM)$, where the superscript T denotes the component tangential to M and $\bar{D}_X Y$ is defined(!) as $\bar{D}_X Y := \bar{D}_{\bar{X}} \bar{Y}$ for any extensions $\bar{X}, \bar{Y} \in \Gamma(T\bar{M})$ of X, Y.

Solution. As we have seen in the lecture (Remark 1.7), $(\bar{D}_{\bar{X}} \bar{Y})_p$ only depends on $\bar{X}_p$ and $\bar{Y} \circ c$, where $c\colon (-\epsilon, \epsilon) \to \bar{M}$ is a curve with $\dot{c}(0) = \bar{X}$. Hence $\bar{D}_X Y$ is independent of the choice of the extensions $\bar{X}$ and $\bar{Y}$.

Clearly, $(\bar{D}_X Y)^{\mathrm{T}}$ defines a linear connection. It remains to prove that this connection is compatible with g and torsion-free. For $X, Y, Z \in TM$, we have

$$\begin{aligned} Zg(X,Y) &= \bar{Z}\bar{g}(\bar{X},\bar{Y}) = \bar{g}(\bar{D}_{\bar{Z}}\bar{X},\bar{Y}) + \bar{g}(\bar{X},\bar{D}_{\bar{Z}}\bar{Y}) \\ &= \bar{g}((\bar{D}_Z X)^{\mathrm{T}},\bar{Y}) + \bar{g}(\bar{X},(\bar{D}_Z Y)^{\mathrm{T}}) = g((\bar{D}_Z X)^{\mathrm{T}},Y) + g(X,(\bar{D}_Z Y)^{\mathrm{T}}) \end{aligned}$$

and

$$(\bar{D}_X Y)^{\mathrm{T}} - (\bar{D}_Y X)^{\mathrm{T}} = (\bar{D}_{\bar{X}}\bar{Y})^{\mathrm{T}} - (\bar{D}_{\bar{Y}}\bar{X})^{\mathrm{T}} = [\bar{X},\bar{Y}]^{\mathrm{T}} = [X,Y].$$

$\square$

2.2. Gradient and Hessian form. Let (M, g) be a Riemannian manifold, D the Levi-Civita connection and $f\colon M \to \mathbb{R}$ a smooth function on M.

(a) The *gradient* $\operatorname{grad} f \in \Gamma(TM)$ is defined by

$$df(X) = g(\operatorname{grad} f, X), \quad \forall X \in \Gamma(TM).$$

Compute $\operatorname{grad} f$ in local coordinates.

(b) The *Hessian form* $\operatorname{Hess}(f) \in \Gamma(T_{0,2}M)$ is defined by

$$\operatorname{Hess}(X,Y) = g(D_X \operatorname{grad} f, Y), \quad \forall X, Y \in \Gamma(TM).$$

Prove that $\operatorname{Hess}(f)$ is symmetric and compute $\operatorname{Hess}(f)$ in local coordinates.

Solution. (a) For a chart (φ, U), let $A_i := \frac{\partial}{\partial \varphi^i}$ and $\operatorname{grad} f = \sum_i Y^i A_i$. Then we have

$$\begin{aligned} f_j &= \tfrac{\partial}{\partial \varphi^j}(f) = df\,(A_j) = g\,(\operatorname{grad} f, A_j) \\ &= g\left(\sum_i Y^i A_i, A_j\right) = \sum_i Y^i g\,(A_i, A_j) = \sum_i Y^i g_{ij} \end{aligned}$$

Hence we get $Y^i = f_j g^{ji}$ and thus $\operatorname{grad} f = \sum_{i,j} g^{ji} f_j A_i$.
(b) First, we use that D is compatible with g. We get

$$\begin{aligned} \operatorname{Hess}(f)(X,Y) &= g(D_X \operatorname{grad} f, Y) = Xg(\operatorname{grad} f, Y) - g(\operatorname{grad} f, D_X Y) \\ &= X(Y(f)) - (D_X Y)(f) \end{aligned}$$

Since the Levi-Civita connection D is torsion free, it follows

$$\begin{aligned} \operatorname{Hess}(f)(X,Y) &= X(Y(f)) - (D_X Y)(f) + T(X,Y)(f) \\ &= Y(X(f)) - (D_Y X)(f) = \operatorname{Hess}(f)(Y,X), \end{aligned}$$

i.e. $\operatorname{Hess}(f)$ is symmetric.

Furthermore, we get in local coordinates

$$\operatorname{Hess}(f)_{ij} = \operatorname{Hess}(f)(A_i, A_j) = A_i(A_j(f)) - (D_{A_i} A_j)(f) = f_{ij} - \sum_k \Gamma^k_{ij} f_k$$

□

2.3. The exponential map for $\mathrm{SO}(n,\mathbb{R})$**.** We consider the matrix group

$$G := \mathrm{SO}(n,\mathbb{R}) = \{g \in \mathbb{R}^{n\times n} : g^{-1} = g^{\mathrm{T}},\ \det(g) = 1\}$$

which acts on $\mathbb{R}^{n\times n}$ by matrix multiplication. Recall that G is a manifold with tangent bundle

$$TG = \{(g, gA) : g \in G,\ A \in \mathbb{R}^{n\times n},\ A^{\mathrm{T}} = -A\}.$$

We say that $X \in \Gamma(TG)$ is a *left-invariant vector field* on G if there is some $X_0 \in TG_e$ such that $X_g = gX_0$ for all $g \in G$.

(a) A Riemannian metric $\langle \cdot, \cdot \rangle$ on TG is called *bi-invariant* if both, left translation $l_g\colon G \to G$, $l_g(x) = gx$ and right tanslation $r_g\colon G \to G$, $r_g(x) = xg$ are isometries.

Prove that

$$\langle (g, gA), (g, gB) \rangle := \tfrac{1}{2}\operatorname{trace}(AB^{\mathrm{T}})$$

defines a bi-invariant Riemannian metric on G.

(b) Let D be the Levi-Civita connection with respect to $\langle\cdot,\cdot\rangle$ on G. Show that for left-invariant vector fields $X, Y \in \Gamma(TG)$ we have

$$D_X(Y) = \tfrac{1}{2}[X,Y].$$

Hint: Use that the Lie-bracket of left invariant vector fields is left-invariant and satisfies $[A,B] = AB - BA$ on TG_e.

(c) Prove that the exponential map $\exp_e \colon TG_e \to G$ is given by

$$\exp_e(A) = e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

Proof. (a) Clearly, this defines a Riemannian metric. We prove bi-invariance. Note that $l_{h*}(g, gA) = (hg, hgA)$ and $r_{h*}(g, gA) = (gh, gAh)$. Thus we have

$$\begin{aligned}
\langle l_{h*}(g,gA), l_{h*}(g,gB)\rangle &= \langle (hg,hgA),(hg,hgB)\rangle \\
&= \tfrac{1}{2}\operatorname{trace}(AB^{\mathrm{T}}) \\
&= \langle (g,gA),(g,gB)\rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle r_{h*}(g,gA), r_{h*}(g,gB)\rangle &= \langle (gh,gAh),(gh,gBh)\rangle \\
&= \langle (gh,gh(h^{-1}Ah)),(gh,gh(h^{-1}Bh))\rangle \\
&= \tfrac{1}{2}\operatorname{trace}(h^{-1}Ah(h^{-1}Bh)^{\mathrm{T}}) \\
&= \tfrac{1}{2}\operatorname{trace}(h^{-1}AB^{\mathrm{T}}h) = \tfrac{1}{2}\operatorname{trace}(AB^{\mathrm{T}}) \\
&= \langle (g,gA),(g,gB)\rangle
\end{aligned}$$

(b) By the Koszul formula, we have

$$\begin{aligned}
2\langle D_X Y, Z\rangle = {}& X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle \\
& - \langle X,[Y,Z]\rangle - \langle Y,[X,Z]\rangle + \langle Z,[X,Y]\rangle.
\end{aligned}$$

First, note that $X\langle Y,Z\rangle = 0$ for left-invariant vector fields $X, Y, Z \in \Gamma(TG)$ as $\langle Y,Z\rangle$ is constant.

Furthermore, it holds

$$\begin{aligned}
-\langle X, [Y, Z]\rangle &= -\tfrac{1}{2}\operatorname{trace}(X_0(Y_0Z_0 - Z_0Y_0)^{\mathrm{T}}) \\
&= -\tfrac{1}{2}\operatorname{trace}(X_0Z_0^{\mathrm{T}}Y_0^{\mathrm{T}}) + \tfrac{1}{2}\operatorname{trace}(X_0Y_0^{\mathrm{T}}Z_0^{\mathrm{T}}) \\
&= -\tfrac{1}{2}\operatorname{trace}(Y_0^{\mathrm{T}}X_0Z_0^{\mathrm{T}}) + \tfrac{1}{2}\operatorname{trace}(Y_0^{\mathrm{T}}Z_0^{\mathrm{T}}X_0) \\
&= -\tfrac{1}{2}\operatorname{trace}((-Y_0)(-X_0)^{\mathrm{T}}Z_0^{\mathrm{T}}) + \tfrac{1}{2}\operatorname{trace}((-Y_0)Z_0^{\mathrm{T}}(-X_0)^{\mathrm{T}}) \\
&= \tfrac{1}{2}\operatorname{trace}(Y_0(X_0Z_0 - Z_0X_0)^{\mathrm{T}}) \\
&= \langle Y, [X, Z]\rangle
\end{aligned}$$

and hence we get $2\langle D_XY, Z\rangle = \langle Z, [X, Y]\rangle$ for all left-invariant vector fields $Z \in \Gamma(TG)$. Therefore $D_X(Y) = \frac{1}{2}[X, Y]$ holds. (c) For $A \in TG_e$, we define a path $c\colon \mathbb{R} \to G$ by $c(t) := e^{tA}$. Note that $c(t) \in G$, since

$$(e^{tA})^{-1} = e^{-tA} = e^{tA^{\mathrm{T}}} = (e^{tA})^{\mathrm{T}}$$

Moreover, we have $\dot{c}(t) = e^{tA}A = c(t)A$ and thus, $\dot{c}$ coincides with the left-invariant vectorfield X determined by A, i.e. $X_g := gA$. Hence, $\frac{D}{dt}\dot{c} = D_XX = \frac{1}{2}[X, X] = 0$ and c is a geodesic with $\dot{c}(0) = A$ and therefore

$$\exp_e(A) = c(1) = e^A.$$

$\square$